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CHARACTERIZATION OF PROJECTIVE INCIDENCE STRUCTURES,(U)

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Characterization of Projective Incidence Structures,

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1. Introduction and statement of theorems

For a finite set X, $|X|$ will denote the number of elements of X. An incidence structure is an ordered triple (P,L,I) where P and L are disjoint sets and $I \subseteq P \times L$. Elements of P will be called points or vertices and elements of L lines. A line ℓ and a point p are called incident iff $(p,\ell) \in I$. We also say in this case that ℓ contains p or p lies on ℓ. Two lines ℓ and m are said to intersect iff they have a common incident point. With any incidence structure (P,L,I) is associated its dual incidence structure (L,P,I^*) where $I^* = \{(\ell,p):(p,\ell) \in I\}$. If L is a set of subsets of P and $(p,\ell) \in I$ iff $p \in \ell$, we will refer to (P,L,I) as $(P,L,\in)$ or (P,L). The dual of $(P,L,\in)$ will be written as $(L,P,\ni)$. If each element of L and P is a set and $(p,\ell) \in I$ iff $p \subseteq \ell$, we write (P,L,I) as $(P,L,\subseteq)$ and its dual as $(L,P,\supseteq)$. For a line ℓ, P_ℓ will denote the set of points incident with line ℓ. If P_ℓ is a finite set, we write $\kappa(\ell)$ for the cardinality of P_ℓ. Similarly, for a point p, L_p denotes the set of lines ℓ incident with the point p and we write $r(p)$ for $|L_p|$. An incidence structure is said to be simple iff for any two distinct lines ℓ and ℓ', $P_\ell \neq P_{\ell'}$. Incidence structures (P,L,I) and (P',L',I') will be called isomorphic iff there exist bijections $\sigma: P \to P'$ and $\tau: L \to L'$ such that $(p,\ell) \in I$ iff $(\sigma(p), \tau(\ell)) \in I'$.

An incidence structure $\pi = (P,L,I)$ is said to be finite iff both P and L are finite sets. All incidence structures in this paper are finite. For a finite incidence structure, we will set

$r(\pi) = \min\{r(p): p \in P\}$ and $k(\pi) = \min\{k(\ell): \ell \in L\}$. Let q be a positive integer. If $q = 1$, we define $s(\pi,q)$ to be equal to $k(\pi)$. If $q \geq 2$, we define $s(\pi,q)$ to be the unique real number s which satisfies $q^s - 1 = k(\pi)(q-1)$. If $q = 1$, we define $d(\pi,q)$ to be equal to $r(\pi) + s(\pi, q) - 1$. If $q \geq 2$, we define $d(\pi,q)$ to be the unique real number d which satisfies $q^{d-s(\pi,q)+1} - 1 = (q-1)r(\pi)$. We normally write $s(\pi,q)$ as $s(\pi)$ and $d(\pi,q)$ as $d(\pi)$. The incidence structure π is said to be semilinear iff $\forall p,p' \in P$, $p \neq p'$, $\exists$ at most one line ℓ incident with both p and p'. Let r and k be positive integers. A semilinear incidence structure π is said to be an (r,k) incidence structure iff for every point p, $r(p) = r$ and every line ℓ, $k(\ell) = k$. Let π be a semilinear incidence structure and ℓ and m be two lines. A line n will be called a transversal of ℓ and m iff n intersects both ℓ and m and $P_n \cap P_\ell \neq P_n \cap P_m$. A semilinear incidence structure π is said to satisfy Pasch's axiom iff for any pair of intersecting lines m_1 and m_2 and any pair of transversals n_1 and n_2 of m_1 and m_2, n_1 intersects n_2. A subset $F \subseteq P$ is called a flat iff $\forall \ell \in L$, $|P_\ell \cap F| \geq 2$ implies $P_\ell \subseteq F$. Clearly, any intersection of flats is a flat. For $S \subseteq P$, the flat $\langle S \rangle = \bigcap_{F \supseteq S} F$ is said to be the flat generated by S. For any flat F, rank F is the smallest integer n such that there exists a set $S \subseteq P$, $|S| = n$ and $\langle S \rangle = F$. The rank of the flat P is called rank π.

A simple graph is a simple incidence structure in which every line is incident with exactly two points. Points and lines of a simple graph will usually be called vertices and edges, respectively. Two vertices p and p' will be called adjacent iff there exists an edge ℓ incident with p and p'. Adjacency is a symmetric relation on the set of vertices of a graph and determines a simple graph completely. All graphs considered in this paper will be finite and simple. Let G be a simple graph with vertex set V and edge set E. Let n be a nonnegative integer. A path of length n from u to v is a sequence $(u = v_0, \ell_1, v_1, \ell_2, v_2, \ldots, \ell_n, v_n = v)$ where ℓ_i is an edge incident with v_{i-1} and v_i, $i = 1,2,\ldots,n$. If the vertices $v_0, v_1, \ldots, v_n$ are all distinct, then the path is said to be simple. If for any two vertices u and v there exists a path from u to v, then the graph G is said to be connected. In a connected graph G the distance $d(u,v)$ between two vertices u and v is the smallest nonnegative integer n such that a path of length n from u to v in G exists.

Let $\pi = (P,L,I)$ be an incidence structure. The adjacency graph $G(\pi)$ of π is a graph having vertex set P and two vertices adjacent iff some line of π contains both. The graph $G(\pi^*)$ of the dual incidence structure π^* will be called the line graph of π. Distance between two points p and p' of π will be same as the distance between them in $G(\pi)$. For $S \subseteq P$ and ℓ, $m \in L$, we will set $d(\ell,S) = \min\{d(p,p')\colon p' \in S,\ p \text{ incident with } \ell\}$ and $d(\ell,m) = \min\{d(\ell,p)\colon p \text{ incident with } m\}$ where $d(p,p')$ is the distance between the vertices p and p' in $G(\pi)$. Sometimes the points of π will be called vertices.

Let $q \geq 2$ be a prime power and $1 \leq s \leq d$ be integers. Let V be a d-dimensional vector space over a finite field of order q. Let W_i be the set of i-dimensional subspaces of V, $1 \leq i \leq d$. Let $(W_{s-1}, W_s, \subseteq)$ be the incidence structure whose points are $(s-1)$-dimensional subspaces, lines are s-dimensional subspaces and incidence is set inclusion. Any incidence structure π isomorphic to $(W_{s-1}, W_s, \subseteq)$ will be called an (s,q,d) projective incidence structure (p.i.s.). For $q = 1$, also we define an $(s,1,d)$-projective incidence structure. Let Y be a finite set with $|Y| = d$. A subset $Y' \subseteq Y$ is called an i-subset of Y iff $|Y'| = i$. Let Z_i be the set of i-subsets of Y. Any incidence structure isomorphic to $(Z_{s-1}, Z_s, \subseteq)$ will be called an $(s,1,d)$-projective incidence structure. The incidence structure $(W_{d-s+1}, W_{d-s}, \supseteq)$ is dual to $(W_{s-1}, W_s, \subseteq)$. Also, $(Z_{d-s+1}, Z_{d-s}, \supseteq)$ is dual to $(Z_{s-1}, Z_s, \subseteq)$.

The following classical theorem about finite projective spaces characterizes $(2,q,d)$-projective incidence structures for $d \geq 4$.

<u>Theorem</u> Let π be a finite incidence structure satisfying

(p1) There exists exactly one line joining two distinct points.

(p2) Every line contains at least three points.

(p3) Pasch's axiom.

(p4) Rank of $\pi \geq 4$.

Then there exists a prime power $q \geq 2$ and an integer $d \geq 4$ such that π is a $(2,q,d)$-projective incidence structure. Conversely, any $(2,q,d)$-projective incidence structure with $d \geq 4$, $q \geq 2$ satisfies (p1) - (p4).

Extending this classical theorem, we prove a characterization of (s,q,d)-projective incidence structures when $3 \leq s < d-1$.

Theorem 1. Let $q \geq 1$ be an integer and π be a finite incidence structure satisfying

(f1) $3 \leq s(\pi,q) < d(\pi,q) - 1$.

(f2) There exists at most one line joining two distinct points.

(f3) If p is a point and ℓ is a line such that $d(p,\ell) = 1$, then there are exactly $(q + 1)$ lines which pass through p and intersect ℓ .

(f4) If p and p' are two distinct points such that $d(p,p') = 2$, then there are exactly $(q + 1)$ lines ℓ such that ℓ passes through p' and $d(p,\ell) = 1$.

(f5) $G(\pi)$ is connected.

Then $s = s(\pi,q)$ and $d = d(\pi,q)$ are integers, $q = 1$ or a prime power and π is an (s,q,d)- projective incidence structure. Conversely, for $3 \leq s < d - 1$, any (s,q,d)-projective incidence structure satisfies (f1) -(f5) .

We also show that the axioms (f1) - (f5) are minimal for the purpose of characterizing (s,q,d)-p.i.s., $3 \leq s < d - 1$. For any choice of $j \in \{1,2,3,4,5\}$, there exists incidence structures π' which satisfy the four axioms other than (f_j) and is not an (s,q,d)-p.i.s. with $3 \leq s < d - 1$. A finite incidence structure π satisfying (f2) - (f5) is called an (s,q,d)-pseudo projective incidence structure where $s(\pi,q) = s$ and $d(\pi,q) = d$. The axiom (f5) in the statement of Theorem 1 is not an essential axiom. Let $\pi_i = (P_i,L_i,I_i)$, $i = 1,2$ be two incidence structures such that $P_1 \cap P_2 = L_1 \cap L_2 = \phi$. We define the direct sum $\pi = \pi_1 + \pi_2$ by

$\pi = (P_1 \cup P_2 , L_1 \cup L_2 , I)$ where $(p,\ell) \in I$ iff $\exists i , 1 \leq i \leq 2$, $p \in P_i$, $\ell \in L_i$, and $(p,\ell) \in I_i$.

Theorem 2. Let $q \geq 1$ be an integer and π be a finite incidence structure satisfying the axioms (f1) - (f4). Then $q = 1$ or a prime power and π is isomorphic to the direct sum of one or more projective incidence structures. Conversely, if $q = 1$ or a prime power and $3 \leq s < d - 1$ and π is the direct sum of several (s_i,q,d_i)-p.i.s. where $3 \leq s_i < d_i - 1$, then π satisfies axioms (f1) - (f4).

Outline of the Proof. Let π be an (s,q,d)-pseudo projective incidence stucture. Let m and n be two lines containing a common point O. A line ℓ is said to be a transversal of m and n iff ℓ intersects both m and n and does not contain O. Let $C(m,n)$ be the set of lines containing the transversals of m and n and all lines ℓ which contain O and intersect at least one transversal of m and n. $C(m,n)$ is called the plane generated by m and n. Let $\mathcal{C}$ be the set of all planes. One of the important steps in the proof is to show that the incidence structure $(L,\mathcal{C},\in)$ is an $(s+1,q,d)$-pseudo projective incidence structure. One starts with an (s,q,d)-pseudo p.i.s. and finally obtains an $(d-1,q,d)$-pseudo p.i.s. which is then shown to be the dual of a projective space.

2. Preliminary propositions

Lemma 1. Let $q \geq 1$ be an integer, π be a finite incidence structure such that $r(p)$ and $k(\ell)$ are positive for all points p and line ℓ. Let π satisfy the axioms (f2),(f3) and (f5) and $r = r(\pi)$, $k = k(\pi)$. Then π is an (r,k)-incidence structure.

Proof: Let $\pi = (P,L,I)$. To show that $\forall \ell \in L$, $k(\ell) = k$, it is sufficient to show that $\forall \ \ell' \in L$, $k(\ell) = k(\ell')$. Let ℓ and ℓ' be two intersecting lines and z be the common point. We calculate

$$b = |\{(p,p') : (p,\ell) \in I , (p',\ell') \in I , p,p' \neq z , d(p,p') = 1\}|$$

For every point $p \neq z$ of ℓ, $d(p,\ell') = 1$. So there are q points p' of ℓ' such that $d(p,p') = 1$ and $p' \neq z$. Hence, $b = (k(\ell)-1)(q)$. By symmetry $b = (k(\ell')-1)(q)$. Since $q \geq 1$, $k(\ell) = k(\ell')$. Let ℓ and ℓ' be any two lines. Since $G(\pi)$ is connected, we can find a sequence $\ell_0 = \ell , \ell_1 , \ell_2, \ldots , \ell_i = \ell'$ such that ℓ_{j-1} and ℓ_j intersect for $j = 1,2,\ldots,i$. Since $k(\ell_{j-1}) = k(\ell_j)$ for $j = 1 , 2 , \ldots , i$, it follows that $k(\ell') = k(\ell)$. It is easily checked that the dual incidence structure π^* satisfies (f2), (f3) and (f5). Therefore, we get $r(p) = r(p')$, $\forall p , p' \in P$ and hence, $r(p) = r$, $\forall p \in P$.

Lemma 2. Let $q = 1$ or a prime power and $3 \leq s < d-1$ be integers. Then any (s,q,d)-projective incidence structure is an (s,q,d)-pseudo projective incidence structure.

Proof: First we consider the case q a prime power, $q \geq 2$. Let $\pi = (W_{s-1}, W_s, \subseteq)$ be an (s,q,d)-projective incidence structure where $3 \leq s \leq d-2$ and W_i is the set of i - dimensional subspaces of a vector space V of dimension d over $GF(q)$, $0 \leq i \leq d$.

The number of $(s-1)$-dimensional subspaces contained in an s-dimensional subspace is $\frac{q^s-1}{q-1}$ and hence, $k(\pi) = \frac{q^s-1}{q-1}$ and $s(\pi) = s$.

Similarly, the number of s-dimensional subspaces containing a given $(s-1)$-dimensional subspace is $\frac{q^{d-s+1}-1}{q-1}$. Therefore, $r(\pi) = \frac{q^{d-s+1}-1}{q-1}$ and $d(\pi) = d$. The axiom (f1) holds since $3 \leq s \leq d-2$. Let p and p' be two $(s-1)$-dimensional subspaces and ℓ be an s-dimensional subspace such that $p, p' \subseteq \ell$. Then ℓ is the subspace spanned by p and p'. Hence, there exists at most one line joining p and p' and π is semilinear. Let p and p' be two $(s-1)$-dimensional subspaces such that $\{u_1, u_2, \ldots, u_i, v_1, \ldots, v_{s-i}\}$ and $\{u_1, u_2, \ldots, u_i, w_1, w_2, \ldots, w_{s-i}\}$ are respectively bases of p and p', $0 \leq i \leq s-1$. Let p_j be the subspace spanned by $\{u_1, u_2, \ldots, u_i, w_1, w_2, \ldots, w_j, v_{j+1}, \ldots v_{s-i}\}$, $j=0,1, \ldots, s-i$. Then $p_0 = p$ and $p_{s-i} = p'$ and p_j and p_{j+1} are adjacent in $G(\pi)$. Hence, there exists a path joining p and p' in $G(\pi)$. This establishes that $G(\pi)$ is connected. Let $p \in W_{s-1}$ and $\ell \in W_s$ such that $d(p,\ell) = 1$. Then $p \not\subseteq \ell$ and there exists an $\ell' \in W_s$ such that $p \subseteq \ell'$ and $\ell \cap \ell' \in W_{s-1}$. It follows that $p \cap \ell = u$ is an $(s-2)$-dimensional subspace. There are $(q + 1)$ $(s-1)$-dimensional subspaces p_i, $1 \leq i \leq q+1$ such that $u \subseteq p_i \subseteq \ell$. Let $\ell_i = \langle p, p_i \rangle$. Then ℓ_i, $1 \leq i \leq q+1$ are the only lines of π which contain p and intersect ℓ in a point. It follows that π satisfies (f3). Let $p, p' \in W_{s-1}$ such that $d(p,p') = 2$. This implies that $p \cap p' = v \in W_{s-3}$.

Let $u_1, u_2, \ldots, u_{q+1}$ be the $(s-2)$-dimensional subspaces such that $v \subseteq u_i \subseteq p$, $1 \le i \le q+1$. Let $\ell_i = \langle u_i, p' \rangle$, $1 \le i \le q+1$. Then $\ell_1, \ell_2, \ldots, \ell_{q+1}$ are the only lines of π which pass through p' and have distance 1 from p. Therefore, π satisfies (f4). This establishes the lemma when $q \ge 2$. For $q = 1$, we take $\pi = (Z_{s-1}, Z_s, \subseteq)$ where Z_i is the set of i-element subsets of a d-set Y, $0 \le i \le d$. It is easily checked that π satisfies the axioms (f1) - (f5).

In the sequel we will assume without loss of generality (wlog) that lines are subsets of points. We assume that q is a fixed positive integer and s and d real numbers satisfying $3 \le s < d-1$ and π is a pseudo projective incidence structure and $s(\pi) = s$, $d(\pi) = d$, $r(\pi) = r$, $k(\pi) = k$.

<u>Lemma 3.</u> Let p and p' be two distinct points of π such that $d(p,p') = 2$. Let L_1 be the set of lines containing p and at distance 1 from p' and let L_2 be the set of lines containing p' and at distance 1 from p. Then each line of L_1 intersects each line of L_2.

<u>Proof:</u> Let $n \in L_2$ and $n^* = \{z \in n : d(z,p) = 1\}$. Then $|n^*|$ equals the number of lines of L_1 which intersect n. By (f3), $|n^*| = (q+1)$ and by (f4), $|L_1| = q+1$. Hence, each line of L_1 intersects n.

For a pair of lines m and n, $T(m,n)$ denotes the set of transversals of m and n.

<u>Lemma 4.</u> Let m and n be two distinct lines of π such that $d(m,n) = 1$. Then (i), $d(p,m) = 1$ for exactly $(q+1)$ points p of n and (ii), $|T(m,n)| \le (q+1)^2$.

Proof: Since $d(m,n) = 1$, there exists points x and y such that $x \in m$, $y \in n$ and $d(x,y) = 1$. Since $d(x,n) = 1$, by (f3) there exists $(q + 1)$ points $y_o = y$, y_1 , ... , y_q such that $d(x,y_i) = 1$, $y_i \in n$ and $d(y_i, m) = 1$, $0 \leq i \leq q$. If possible, let $y \in n$, $y \neq y_i$, $0 \leq i \leq q$ and $d(y,m) = 1$. Then $d(x,y) = 2$, $d(x,n) = d(y,m) = 1$ and $x \in m$, $y \in n$. By Lemma 4, m and n must intersect whence $d(m,n) \neq 1$. This completes the proof of (i) and (ii) follows easily.

Let m and n be intersecting lines and x be the point of intersection. We let

$$C(m,n) = T(m,n) \cup \{h: h \in L , x \in h , h \cap n' \neq \phi \text{ for some } n' \in T(m,n)\}$$

Lemma 5. Pasch's axiom is valid in (P,L).

For any pair of intersecting lines m_1 and m_2 , $|T(m_1,m_2)| = (k-1)q$.

Proof: Let $\{x\} = m_1 \cap m_2$. For each $y \in m_1 - x$, y is adjacent to q vertices of $m_2 - x$. So, q transversals of m_1 and m_2 contain y . Therefore, $|T(m_1,m_2)| = (k-1)q$. Let $n \in T(m_1,m_2)$. Let $a \in n \cap m_1$, $b \in n \cap m_2$, S_1 be the set of $(q-1)$ vertices of $m_1 - \{x,a\}$ adjacent to b and S_2 be the set of $(q-1)$ vertices of $m_2 - \{x,b\}$ adjacent to a . Let $h \in T(m_1,m_2)$ such that $\{c\} = h \cap m_1 \notin S_1$. Then b and c are not adjacent. We get $d(b,c) = 2$, $b \in n$, $d(c,n) = 1$, $c \in h$, $d(b,h) = 1$. By Lemma 4, n and h intersect. It follows that if $h \in T(m_1,m_2)$ and h and n do not intersect, then $h \cap m_1 \in S_1$. Similarly, $h \cap m_2 \in S_2$. Therefore the number of lines of $T(m_1,m_2)$ not intersecting n is at most $(q-1)^2 = |S_1| \, |S_2|$. If $q = 1$, $(q-1)^2 = 0$. Then all lines of $T(m_1,m_2)$ intersect n , so Pasch's axiom is valid.

Let $q \geq 2$. If possible, let n and h be two non-intersecting lines of $T(m_1,m_2)$. There are at least $|T(m_1,m_2)|-2(q-1)^2 = (k-1)q-2(q-1)^2$ lines of $T(m_1,m_2)$ which intersect both n and h . Also, m_1 and m_2 intersect both n and h . Hence, the number of lines intersecting both n and h is at least $kq - 2q^2 + 3q$. On the other hand, since $d(n,h) = 1$, by Lemma 5 there are at most $(q + 1)^2$ lines intersecting both n and h . This gives us $(q + 1)^2 \geq kq - 2q^2 + 3q$. Since $s \geq 3$, $k \geq q^2 + q + 1 \geq 3q$ and $(q+1)^2 \geq 3q^2 - 2q^2 + 3q$. Simplifying the inequality we get $1 \geq q$ which contradicts the assumption.

If S is a set of lines such that any two lines of S intersect each other, then S is a clique in the line graph of (P,L); we refer to such a set S as a clique of lines.

Lemma 6. Let m_1 and m_2 be intersecting lines. Then $C(m_1,m_2)$ is a maximal clique of lines.

Proof: We denote $T(m_1,m_2)$ by T and $C(m_1,m_2)$ by C . Let $\{x\} = m_1 \cap m_2$. T is a clique of lines, and so is $C-T$ since x belongs to each line of $C-T$. It is sufficient to show that if $h \in C-T$ and $n' \in T$, then h intersects n' . Since $h \in C-T$, $x \in h$ and h intersects n for some transversal n of m_1 and m_2 . We may assume (by exchanging m_1 and m_2 if necessary) that $n \cap m_2 \neq n' \cap m_2$. Then $h,n' \in T(n,m_2)$. So, h and n' intersect. Hence C is a clique of lines. It is clear from the definition of C that no proper superset of C is a clique of lines.

We call each $C(m_1,m_2)$ a plane, and let $\mathcal{C}$ be the set of planes.

Corollary 1. Each plane contains $qk + 1$ lines.

Proof: Let m and n be lines which intersect at x . We show that $|C(m,n)| = qk + 1$. Let $h \in T(m,n)$. By Lemma 6 every line of $C(m,n)-T(m,n)$ intersects h, so $C(m,n)-T(m,n)$ is the set of $q+1$ lines which contain x and intersect h. $|T(m,n)| = (k-1)q$.

Lemma 7. Let K be a clique of lines, $m,n \in K$ $(m \neq n)$, and $\{x\} = m \cap n$. Then either all lines of K contains x or $K \subseteq C(m,n)$.

Proof: We assume that some line n' of K does not contain x and show that $K \subseteq C(m,n)$. Let $h \in K$. Then h intersects m, n, and n'. If $x \notin h$, then $h \in T(m,n)$. So $h \in C(m,n)$. Next suppose $x \in h$. Since h intersects n', $h \in C(m,n)$. Therefore, $K \subseteq C(m,n)$.

Lemma 8. (i) Each pair of intersecting lines is in a unique plane. (ii) If the plane C contains at least 1 line containing x, then C contains exactly $q+1$ lines containing x. (iii) Each line is contained in $(r-1)/q$ planes.

Proof: (i) Let m and n be intersecting lines and the plane C contain m and n. By Lemma 7 $C \subseteq C(m,n)$. But all planes have the same cardinality, so $C = C(m,n)$. (ii) Let $x \in m \in C$. Let $n \in C$ so that $x \notin n$. Then $C = C(m,n)$. Every line of C which contains x also intersects n. There are $q+1$ lines which contain x and intersect n. One of these lines is m, and the remaining q lines are transversals of m and n, so $q+1$ lines of C contain x. (iii) Let m be a line. Choose $x \in m$ and let $m_2, m_3, \ldots, m_r$ be the lines containing x which are distinct from m. Each plane which contains m contains exactly q lines among $m_2, m_3, \ldots, m_r$. By part (i), each line m_i is contained in a unique plane containing m. Hence exactly $(r-1)/q$ planes contain m.

From Lemma 8 and Corollary 1, the following statement is immediate.

<u>Corollary 2.</u> $(L,\mathcal{C})$ is a semilinear $((r-\mathcal{C}/q\ ,\ qk+1)$ - incidence structure.

For any plane C we define $\overline{C} = \bigcup_{m \in C} m$.

<u>Lemma 9.</u> Let $m \in L$ and $C \in \mathcal{C}$. If $|m \cap \overline{C}| \geq 2$, then $m \in C$.

<u>Proof:</u> Let $x,y \in m \cap \overline{C}$. Then for some n_1 and n_2 $(n_1,n_2 \neq m)$ $x \in n_1 \in C$ and $y \in n_2 \in C$. Lines n_1 and n_2 intersect since all lines of C intersect, so $C = C(n_1,n_2)$. Since m is a transversal of n_1 and n_2 , $m \in C$.

Since each pair of intersecting lines is contained in a plane, and each plane is a clique of lines, two lines contain a point in common iff they are both contained in some plane. Therefore the adjacency graph of $(L,\mathcal{C})$ is identical to the line graph of (P,L) . Let H be the adjacency graph of $(L,\mathcal{C})$.

<u>Lemma 10.</u> If m and n are distinct lines then $d_H(m,n) = d_G(m,n) + 1$. If the line m is not contained in the plane C then $d_H(m,C) = d_G(m,\overline{C}) + 1$.

<u>Proof:</u> Let $d_G(m,n) = i-1$ where $m \neq n$. Denote m by m_0 and n by m_i . Let $(m_0,x_1,m_1,x_2,\ldots,x_i,m_i)$ be a sequence of points and lines such that x_j is contained in m_{j-1} and m_j $(1 \leq j \leq i)$. Let $C_j = C(m_{j-1},m_j)$ for $1 \leq j \leq i$. Then $(m_0,C_1,m_1,C_2,\ldots,C_i,m_i)$ is a sequence of lines and planes so that C_j contains m_{j-1} and m_j $(1 \leq j \leq i)$, so $d_H(m,n) \leq i$. Since the direction of this argument is reversible, we may conclude that $d_H(m,n) = d_G(m,n) + 1$.

Let $m \notin C$. Now $d_G(m,\overline{C}) = \min\{d_G(m,n) : n \in C\}$ and $d_H(m,C) = \min\{d_H(m,n) : n \in C\}$. Since $d_H(m,n) = d_G(m,n) + 1$ for distinct lines m and n , $d_H(m,C) = d_G(m,\overline{C}) + 1$.

Lemma 11. $(L,\mathcal{C})$ is an $(s+1,q,d)$-pseudo projective incidence structure.

Proof: We have already established that $(L,\mathcal{C})$ is a semilinear (r^*,k^*)-incidence structure where $r^* = (r-1)/q = (q^{d-s}-1)/(q-1)$ and $k^* = qk+1 = (q^{s+1}-1)/(q-1)$. (If $q = 1$ then $r^* = (r-1)/q = d-s$ and $k^* = qk+1 = s+1$.) The graph H is connected since G is. We prove (f4). Let m and n be lines and $d_H(m,n) = 2$ (so $d_G(m,n) = 1$). Let $S = \{C : C \in \mathcal{C}, n \in \mathcal{C}, d_H(m,C) = 1\}$. We are to show $|S| = q + 1$. Now $S = \{C : C \in \mathcal{C}, n \in \mathcal{C}, m \cap \overline{C} \neq \emptyset\}$.

If h is a line and z a vertex so that $d_G(z,h) = 1$ then there exists at least two lines h_1 and h_2 so that $z \in h_i$ and h_i intersects h $(i = 1,2)$. The plane $C(h_1,h_2)$ contains both h and z. For any plane C containing both z and h we have $|h_i \cap \overline{C}| \geq 2$ so $h_i \in C$ $(i = 1,2)$, and consequently $C = C(h_1,h_2)$. Therefore for any line h and vertex z so that $d_G(z,h) = 1$, a unique plane contains both z and h.

Lines m and n do not intersect. So, no plane contains both. Every plane S contains n and at least one point of m. Let $x_o,x_1,\ldots,x_q$ be the points of m satisfying $d_G(x_i,n) = 1$ $(0 \leq i \leq q)$. Let C_i be the unique plane containing x_i and n $(0 \leq i \leq q)$. If for some i and j $(i \neq j)$ $C_i = C_j$ then $|m \cap \overline{C}_i| \geq 2$. By Lemma 9 this would imply that $m \in C_i$, which is false. Then $S = \{C_o,C_1,\ldots,C_q\}$, so $|S| = q + 1$.

To prove (f3), let $d_H(m,C) = 1$. Then $d_G(m,C) = 0$. So, $m \cap \overline{C} \neq \emptyset$. By Lemma 9, $|m \cap \overline{C}| = 1$. Let $\{x\} = m \cap \overline{C}$.

We are to show that $d_H(m,n) = 1$ for exactly $q + 1$ lines n of C. In other words $d_G(m,n) = 0$ for exactly $q + 1$ lines n of C. But this is clear, since exactly $q + 1$ lines of C contain x.

Lemma 12. Pasch's axiom is valid in $(\mathcal{C},L,\ni)$.

Proof: We first state Pasch's axiom for $(\mathcal{C},L,\ni)$, recalling that two lines intersect (i.e. contain a vertex in commom) iff they are both incident with some plane. Pasch's axiom for $(\mathcal{C},L,\ni)$ states that if lines m and n intersect, and lines h_1 and h_2 intersect both m and n but no plane contains h_1, m, and n and no plane contains h_2, m, and n, then h_1 and h_2 intersect.

Let $\{x\} = m \cap n$. Now $h_1 \notin C(m,n)$, so $h_1 \notin T(m,n)$. Since $h_1 \notin T(m,n)$ but h_1 intersects both m and n, $x \in h_1$. Similarly $x \in h_2$. Therefore h_1 and h_2 intersect, and Pasch's axiom is valid.

Let $\overline{P} = \{\overline{P} : p \in P\}$ where $\overline{p} = \{p : p \in L, p \in m\}$.

Lemma 13. The mapping $\alpha : P \to \overline{P}$ defined by $\alpha(p) = \overline{p}$ is a bijection.

Proof: The mapping α is clearly surjective. We show that α is injective. (P,L) is a semilinear (r,k)-incidence structure, therefore $|\overline{x}| = r > 1$ and $|\overline{x} \cap \overline{y}| \leq 1$ for all $x,y \in P$. It follows that $\overline{x} \neq \overline{y}$ for all distinct $x,y \in P$.

Lemma 14. $\overline{P} \cup \mathcal{C}$ is a partition of the set of maximal cliques of H .

Proof: It is clear from Lemma 7 that every maximal clique of lines is contained in $\overline{P} \cup \mathcal{C}$. We have shown that every plane is a maximal clique of lines. Therefore it is sufficient to show that $\overline{x}$ is a maximal clique of lines for every $x \in P$, and that $\overline{X}$ and $\mathcal{C}$ are disjoint. $\overline{X}$ and $\mathcal{C}$ are disjoint because the lines of a plane are not concurrent.

Let $x \in P$. Clearly $\overline{x}$ is a clique of lines. Let K be a maximal clique containing $\overline{x}$. If possible, let $K \supsetneq \overline{x}$. Let $m \in K - \overline{x}$. Then $x \notin m$. By (f3), the number of lines of $\overline{x}$ intersecting m is at most $q + 1$. Since K is a clique of lines, every line of $\overline{x}$ intersects m. Therefore $q + 1 \geq |\overline{x}| = r > q + 1$ which is a contradiction.

In Lemmas 15 - 17 we examine (s,q,d)-pseudo projective incidence structures where $s = d - 1$.

Lemma 15. Let (P,L) be a $(d-1,q,d)$-pseudo projective incidence structure. Then any two lines intersect and if $q = 1$, $|L| = d$.

Proof: (P,L) is an (r,k)-incidence structure where $r = q + 1$ and $k = (q^{d-1}-1)/(q - 1)$ (if $q = 1$ then $k = d - 1$).

Let G be the adjacency graph of (P,L). Since G is connected, the distance between any two lines is finite. If not all lines intersect then there are lines m and n so that $d(m,n) = 1$. Assume that $d(m,n) = 1$. Then for some $x \in m, d(x,n) = 1$. By (f3) $q + 1$ lines contain x and intersect n. Then these lines together with m constitute $q + 2$ lines containing x, which violates the condition $r = q + 1$. Therefore any two lines intersect.

Let $m \in L$. Since $k(r-1)$ lines intersect m and all lines intersect, $|L| = k(r-1) + 1$. If $q = 1$, $k = d-1$ and $|L| = d$.

Lemma 16. Let (P,L) be a $(d-1,q,d)$-pseudo projective incidence structure where $d > 3$ and $q \geq 2$, and let the incidence structure dual to (P,L) satisfy Pasch's axiom. Then

(i) q is a prime power and d is an integer,

(ii) the incidence structure dual to (P,L) is a $(2,q,d)$-

projective incidence structure,

and (iii) (P,L) is a $(d-1,q,d)$-projective incidence structure.

Proof: For $r = q + 1$ and some k, (P,L) is an (r,k)-incidence structure.

We show that $(L,P,\ni)$ satisfies the axioms (p1) - (p4) of section 1. Now elements of L will be called points and elements of P will be called lines. By Lemma 15 any two points are incident with some line. Therefore $(L,P,\ni)$ satisfies (p1). By hypothesis (p3) is satisfied. Every element of P is incident with $q + 1 \geq 3$ elements of L. Since $d > 3$ every element of L is incident with more than $q + 1$ elements of P. It easily follows that rank of (P,L) is at least 4. Therefore by the theorem about finite projective spaces $(L,P,\in)$ is a $(2,q',d')$-projective incidence structure. Clearly we must have $q' = q$ and $d' = d$. This establishes (ii) and (i). Since a $(d-1,q,d)$-projective incidence structure is dual to a $(2,q,d)$-projective incidence structure (iii) follows.

Lemma 17. Let (P,L) be a $(d-1,1,d)$-pseudo projective incidence structure where $d > 2$. Then d is an integer and (P,L) is a $(d-1,1,d)$-projective incidence structure.

Proof: (P,L) is an (r,k)-incidence structure with $r = 2$ and $k = d-1$. Since k is an integer, d is an integer. We examine the dual incidence stucture $(L,P,\in)$. Elements of L will be called dual points and elements of P dual lines. Each dual line is incident with exactly 2 dual points. Therefore dual lines are equivalent to the edges of the adjacency graph of $(L,P,\in)$. By Lemma 15, each pair of dual points is incident with some dual line.

So, the adjacency graph of $(L,P,\in)$ is the complete graph on $|L| = d$ vertices. Let Y be a d - set and Z_i be the set of i-subsets of Y, $1 \leq i \leq d-1$. We have proved that $(L,P,\in)$ is isomorphic to (Y,Z_2). Therefore (P,L) is isomorphic to $(Z_2,Y,\in)$ and hence to $(Z_{d-2},Z_{d-1},\subseteq)$.

Lemma 18. There is no (s,q,d)-pseudo projective incidence structure where $3 \leq s$ and $d-2 < s < d-1$.

Proof: Assume $\pi = (P,L)$ is an (s,q,d)-pseudo projective incidence structure where $3 \leq s$ and $d-2 < s < d-1$. If $q = 1$ then $r(\pi) = d-s+1$ is not an integer. Therefore $q > 1$. Define $\mathcal{C}$ as in Lemmas 6 - 11. By Lemma 11 $\pi^* = (L,\mathcal{C})$ is an $(s+1,q,d)$-pseudo projective incidence strucutre. $r(\pi^*) = (q^{d-s}-1)/(q-1)$ so $1 < r(\pi^*) < q+1$. Since $r(\pi^*) \geq 2$ and $k(\pi^*) \geq 2$ there exist $m \in L$ and $C \in \mathcal{C}$ so that in the adjacency graph of π^* $d(m,C) = 1$. By (f3), $r(m) \geq q+1$. Since $r(m) = r(\pi^*)$ the impossibility of the assumed incidence structure is established.

3. Proof of the Theorems.

The heart of the inductive procedure for Theorem 1 is contained in the next lemma.

Lemma 19. For $j = 1, 2$ let

(i) B_j be a set,

(ii) A_j and C_j be sets of subsets of B_j,

(iii) the incidence structures (B_j, A_j) and (B_j, C_j) have the same adjacency graph H_j,

(iv) $A_j \cup C_j$ be the set of maximal cliques of H_j,

(v) $A_j \cap C_j = \emptyset$.

Let (B_1, C_1) and (B_2, C_2) be isomorphic. Then $(A_1, B_1, \ni)$ and $(A_2, B_2, \ni)$ are isomorphic.

Proof. By hypothesis (B_1, C_1) and (B_2, C_2) are isomorphic; let $\sigma\colon B_1 \to B_2$ and $\tau\colon C_1 \to C_2$ be bijections which preserve incidence. For any $B' \subseteq B_1$ we let $\sigma(B') = \{\sigma(b): b \in B'\}$; in particular, for $c \in C_1$, $\sigma(c) = \{\sigma(b); b \in c\}$. Then $\sigma(c) = \tau(c)$ for all $c \in C_1$.

σ is an isomorphism between the adjacency graph H_1 of (B_1, C_1) and the adjacency graph H_2 of (B_2, C_2). Therefore σ induces a bijection between the maximal cliques of H_1 and the maximal cliques of H_2. The set of maximal cliques of H_1 is $A_1 \cup C_1$ and the set of maximal cliques of H_2 is $A_2 \cup C_2$. Since $A_1 \cap C_1 = \emptyset = A_2 \cap C_2$ and σ induces a bijection from C_1 to C_2, σ induces a bijection from A_1 to A_2. Then the bijection σ: $B_1 \to B_2$ and the bijection from A_1 to A_2 induced by σ show that the incidence structures (B_1, A_1) and (B_2, A_2) are isomorphic, and also that $(A_1, B_1, \ni)$ and $(A_2, B_2, \ni)$ are isomorphic.

In order to shorten the proof of Theorem 1, we introduce some terminology. For $q = 1$ and a positive integer d, $V_{d,q}$ will denote a finite d-element set. For q a prime power $V_{d,q}$ will denote a d-dimensional vectorspace over $GF(q)$. For $q = 1$ an i dimensional object of $V_{d,q}$ will mean an i-element subset of $V_{d,q}$. For q a prime power, an i-dimensional object of $V_{d,q}$ will mean an i-dimensional subspace of $V_{d,q}$. For $0 \leq i \leq d$, W_i will denote the set of i-dimensional objects of $V_{d,q}$.

Proof of Theorem 1. Assume that there exists a counter example to the statement of Theorem 1. Among all such counter examples we choose an incidence structure $\pi = (P, L, I)$ for which $r(\pi)$ is as small as possible. Wlog we assume that lines are subsets of points. We write s for $s(\pi)$ and d for $d(\pi)$. Let $\mathcal{C}$ be as in Section 2. By Lemma 11, $\pi^* = (L, \mathcal{C})$ is an $(s + 1, q, d)$ - pseudo projective incidence structure. Note that $r(\pi^*) < r(\pi)$ and that the dual of π^* satisfies Pasch's axiom by Lemma 12. By Lemma 18, $s \leq d - 2$. If $s < d - 2$, then π^* satisfies the hypotheses of the theorem and $r(\pi^*) < r(\pi)$. Therefore π^* is an $(s + 1, q, d)$ - projective incidence structure. If $s = d - 2$, then by Lemmas 16 and 17 π^* is an $(d - 1, q, d)$ - projective incidence structure. So in either case d is an integer, $q = 1$ or is a prime power and π^* is isomorphic to $(W_s, W_{s+1}, \subseteq)$ where W_i is the class of i dimensional objects of a $V_{d,q}$ where $i = s, s + 1$.

For $w \in W_{s+1}$, let $\overline{w} = \{u\colon\ u \in W_s \text{ and } u \subseteq w\}$ and $\overline{W}_{s+1} = \{\overline{w}\colon\ w \in W_{s+1}\}$.

For $w \in W_{s-1}$, let $w' = \{u\colon\ u \in W_s, u \supseteq w\}$ and $W'_{s-1} = \{w'\colon\ w \in W_{s-1}\}$.

It is easily seen that $(W_s, W_{s+1}, \subseteq)$ is isomorphic to $(W_s, \overline{W}_{s+1})$ and $(W_{s-1}, W_s, \subseteq)$ is isomorphic to $(W_{s-1}, W'_s, \ni)$. We now apply Lemma 19 with $B_1 = W_s$, $C_1 = \overline{W}_{s+1}$ and $A_1 = W'_{s-1}$, $B_2 = L$, $C_2 = \mathcal{C}$ and $A_2 = \overline{P}$. $(W_s, \overline{W}_{s+1})$ and (W_s, W'_{s-1}) have the same adjacency graph H_1. $W'_{s-1} \cup \overline{W}_{s+1}$ is a partition of the set of maximal cliques of H_1. By the remark after Lemma 9, $(L, \mathcal{C})$ and $(L, \overline{P})$ have the same adjacency graph H_2. By Lemma 14, $\overline{P} \cup \mathcal{C}$ is a partition of the set of maximal cliques of H_2. Finally $(L, \mathcal{C})$ and $(W_s, \overline{W}_{s+1})$ are isomorphic. Therefore by Lemma 19 $(\overline{P}, L, \ni)$ and $(W'_{s-1}, W_s, \ni)$ are isomorphic and hence (P, L, I) and $(W_{s-1}, W_s, \subseteq)$ are isomorphic. Hence there is no counter example to the statement of Theorem 1.

<u>Proof of Theorem 2</u>. Wlog assume that lines of π are subsets of points. Consider the connected components of $G(\pi)$. Let P_1 be

the vertex set of the ith component, $1 \leq i \leq t$. Let L_i be the set of lines of π which contain at least one point of P_i, $1 \leq i \leq t$. Then $P = P_1 \cup P_2 \cup \ldots \cup P_t$ and $L = L_1 \cup L_2 \cup \ldots \cup L_t$ are partitions and each line of L_i is a subset of P_i, $1 \leq i \leq t$. It is easily checked that for $1 \leq i \leq t$, (P_i, L_i) satisfy the axioms (f1) - (f5) with respect to the integer q. Therefore for some integers s_i and d_i (P_i, L_i) is an (s_i, q, d_i)-projective incidence structure and π is the direct sum of these incidence structure. The converse follows from Lemma 2.

4. Minimality of the Axioms.

Let $\mathcal{P}$ be the class of (s, q, d) - projective incidence structure with $3 \leq s \leq d - 2$. The axioms (f1) - (f5) form a minimal set of axioms for the purpose of characterization of the class $\mathcal{P}$. We now demonstrate the minimality of the axiom set (f1) - (f5). For $j \in \{1, 2, 3, 4, 5\}$ we choose q and construct an incidence structure π' which satisfies the four axioms other than (fj) and is not a member of $\mathcal{P}$. For $j = 5$, we saw that the direct sum of two (s, q, d) - projective incidence structures $(3 \leq s \leq d - 2)$ satisfy the four axioms other than (f5). For $j = 1$, our example is a nondesarguesian finite projective plane π of order q. It is easy to see that π satisfies the axioms (f2) - (f5) and that $s(\pi) = 2$. Since π is non desarguesian, π is not an (s, q, d) - projective incidence structure. For $j = 2$, we construct an example as follows. Let q be a given prime power. We choose positive integers s and d satisfying $3 \leq s \leq d - 2$ and $(2q+1)^2 + (2q+1) + 1 \leq \mathrm{Min}(\frac{q^s-1}{q-1}, 2\frac{(q^{d-s+1}-1)}{q-1})$.

Let π be an (s, q, d) - projective incidence structure. The point set of the incidence structure π' will be same as that of π and for each line ℓ of π, π' will have two lines ℓ and ℓ' with $P_\ell = P_{\ell'}$. It is easily checked $k(\pi') = \frac{q^s-1}{q-1}$ and $r(\pi') = 2\frac{(q^{d-s+1}-1)}{q-1}$. Therefore with respect to $(2q+1)$, $3 \leq s(\pi') \leq d(\pi') - 2$ and also π' satisfies (f3), (f4) and (f5) w.r.t. $2q+1$. Clearly π' is not a member of $\mathcal{P}$ and is not an (s, q, d) - projective incidence structure. For $j = 3$, we proceed to construct an example as follows. Consider an affine space

$\mathrm{Aff}(n, q)$ where q is a prime power, and $q^2 + q + 1 \leq \frac{q^{n-2}-1}{q-1}$. Let π' be an incidence structure whose points are the planes of the affine space and lines are the 3-spaces of the affine space and incidence is containment. Points and lines of π will be respectively called ideal points and ideal lines. It will be helpful to view the affine space as a projective space $PG(n, q)$ minus a hyperplane Σ. The number of planes contained in an affine 3-space is $q^3 + q^2 + q$. Therefore $k(\pi) = q^3 + q^2 + q$ and $r(\pi) = \frac{q^{n-2}-1}{q-1}$. For an ideal point p and an ideal line ℓ, p' and ℓ' will respectively denote the corresponding projective plane and projective 3-space. The axiom (f1) is satisfied by π'. Clearly (f2) and (f5) hold for π'. Let p_1 and p_2 be two ideal points such that $d(p_1, p_2) = 2$. Then p_1 and p_2 are affine planes (Figure 1). Since $d(p_1, p_2) = 2$, there exist an ideal point p_3 such that $d(p_i, p_3) = 1$, $i = 1, 2$. Hence $\langle p_i', p_3' \rangle$ is a 3-space for $i = 1, 2$. Therefore $p_i' \cap p_3'$ is a line for $i = 1, 2$. Therefore $p_1' \cap p_2'$ is a point O. Let ℓ be an ideal line such that p_1 is incident with ℓ and $d(p_2, \ell) = 1$. Then ℓ' is a projective 3-space such that $p_1' \subseteq \ell'$ and $p_2' \cap \ell'$ is a projective line passing through O. Let $x_i (i = 0, 1, \ldots, q)$ be the projective lines of p_2' passing through O. Then letting $\ell_i' = \langle p_1', x_0 \rangle$, $0 \leq i \leq q$, ℓ_i' $0 \leq i \leq q$ are all the projective 3-spaces satisfying $p_1' \subseteq \ell'$, $p_2' \cap \ell' =$ a projective line. The corresponding affine 3-spaces ℓ_i, $0 \leq i \leq q$ are all the ideal lines satisfying $d(p_1, \ell) = 0$ and $d(p_2, \ell) = 1$. We proved that π' satisfies (f4) w.r.t. q.

Aff(n, q) where q is a prime power, and $q^2 + q + 1 \leq \frac{q^{n-2}-1}{q-1}$. Let π' be an incidence structure whose points are the planes of the affine space and lines are the 3-spaces of the affine space and incidence is containment. Points and lines of π will be respectively called ideal points and ideal lines. It will be helpful to view the affine space as a projective space PG(n, q) minus a hyperplane Σ. The number of planes contained in an affine 3-space is $q^3 + q^2 + q$. Therefore $k(\pi) = q^3 + q^2 + q$ and $r(\pi) = \frac{q^{n-2}-1}{q-1}$. For an ideal point p and an ideal line ℓ, p' and ℓ' will respectively denote the corresponding projective plane and projective 3-space. The axiom (f1) is satisfied by π'. Clearly (f2) and (f5) hold for π'. Let p_1 and p_2 be two ideal points such that $d(p_1, p_2) = 2$. Then p_1 and p_2 are affine planes (Figure 1). Since $d(p_1, p_2) = 2$, there exist an ideal point p_3 such that $d(p_i, p_3) = 1$, $i = 1, 2$. Hence $\langle p_i', p_3' \rangle$ is a 3-space for $i = 1, 2$. Therefore $p_i' \cap p_3'$ is a line for $i = 1, 2$. Therefore $p_1' \cap p_2'$ is a point O. Let ℓ be an ideal line such that p_1 is incident with ℓ and $d(p_2, \ell) = 1$. Then ℓ' is a projective 3-space such that $p_1' \subseteq \ell'$ and $p_2' \cap \ell'$ is a projective line passing through O. Let $x_i (i = 0, 1, \ldots, q)$ be the projective lines of p_2' passing through O. Then letting $\ell_i' = \langle p_1', x_0 \rangle$, $0 \leq i \leq q$, ℓ_i' $0 \leq i \leq q$ are all the projective 3-spaces satisfying $p_1' \subseteq \ell'$, $p_2' \cap \ell' =$ a projective line. The corresponding affine 3-spaces ℓ_i, $0 \leq i \leq q$ are all the ideal lines satisfying $d(p_1, \ell) = 0$ and $d(p_2, \ell) = 1$. We proved that π' satisfies (f4) w.r.t. q.

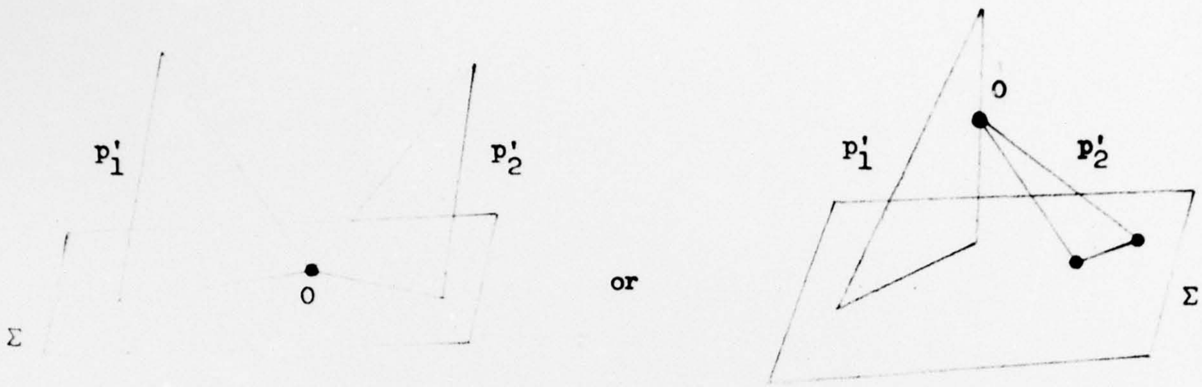

Figure 1

We now show that (f3) does not hold in π'. Let ℓ be an ideal line and p be an ideal point such that $d(p, \ell) = 1$. Let p' and ℓ' be the corresponding projective plane and 3-space respectively. Since $d(p, \ell) = 1$, $p' \cap \ell'$ must be a projective line. Case 1.(Figure 2) $p' \cap \ell' = y$ is a projective line contained in Σ. There are $(q + 1)$ planes of ℓ' which contain y. Of these one is $\ell' \cap \Sigma$ which does not correspond to an ideal point of π. Therefore in case 1 there are q ideal points p_i such that p_i is incident with ℓ and $d(p, p_i) = 1$, $1 \leq i \leq q$.

Case 2.(Figure 3) $p' \cap \ell' = y$ is a projective line not contained in Σ. In this case there will be $(q + 1)$ ideal points p_i such that p_i is incident with ℓ and $d(p, p_i) = 1$ $0 \leq i \leq q$. With respect to q, π' satisfies all the four axioms except (f3) and $\pi' \notin P$.

Figure 2

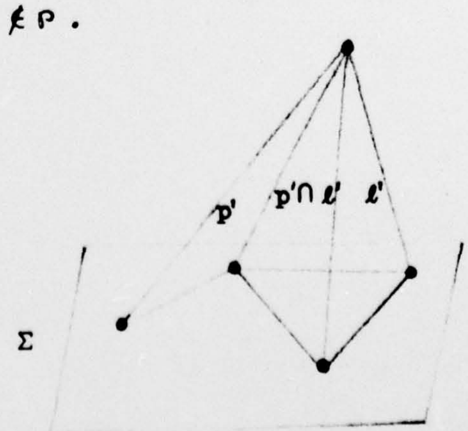

Figure 3

We now consider $j = 4$. Let q be a prime power, $n \geq 4$, $PG(n, q)$ be an n-dimensional projective space over $GF(q)$ and Σ_{n-3} be an $(n-3)$-flat of $PG(n, q)$. Let π' be an incidence structure whose points are the lines of $PG(n, q)$ not intersecting Σ_{n-3} and lines are the planes of $PG(n, q)$ not intersecting Σ_{n-3}. As before points and lines of π will be referred to as ideal points and ideal lines respectively. Lines and planes of $PG(n, q)$ will be called projective lines and projective planes. Clearly every ideal line is incident with $q^2 + q + 1$ ideal point and hence $k(\pi) = q^2 + q + 1$. The number of projective planes of $PG(n, q)$ containing a given projective line is $\frac{q^{n-1}-1}{q-1}$. Of these projective planes $\frac{q^{n-2}-1}{q-1}$ will intersect Σ_{n-3}. Hence the number of ideal lines passing through a given ideal point is q^{n-2}. Since $n \geq 4$, the axiom (f1) holds for π' with respect to $(q - 1)$.

Clearly (f2) and (f5) hold for π'. We now check (f3) for π. Let p and ℓ respectively be an ideal point and an ideal line such that $d(p, \ell) = 1$. Then the projective line p intersects the projective plane ℓ in a projective point O (Figure 4). Let $\Sigma_{n-1} = \langle p, \Sigma_{n-3} \rangle$ be the span of p and Σ_{n-3} and $p_0 = \ell \cap \Sigma_{n-1}$. There are $q + 1$ projective lines of ℓ passing through O. Let $p_0, p_1, \ldots, p_q$ be these lines. The projective plane $\langle p, p_0 \rangle$ is contained in Σ_{n-1}, intersects Σ_{n-3} and hence is not an ideal line of π. Therefore the distance between the ideal points p and p_0 is greater than 1. The ideal points

$p_1, p_2, \ldots, p_q$ are the only ideal points p satisfying $p_i \subset \ell$ and $d(p, p_i) = 1$, $1 \leq i \leq q$. Therefore π satisfies (f3) with respect to $(q - 1)$.

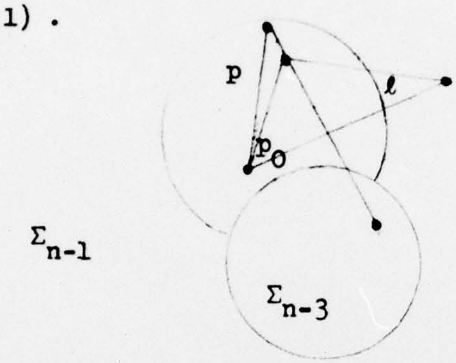

Figure 4

We now show that (f4) does not hold in π'. Let p_1 be an ideal point and $\langle \Sigma_{n-3}, p_1 \rangle = \Sigma_{n-1}$. Let p_2 be an ideal point such that $d(p_1, p_2) = 2$. Case 1. p_2 is a projective line not intersecting p_1 and Σ_{n-3} and not contained in Σ_{n-1} (Figure 5). Let $x_i, 0 \leq i \leq q$ be the $q + 1$ points of p_2 where $x_0 \in \Sigma_{n-1}$, and $\ell_i = \langle p_1, x_i \rangle$, $0 \leq i \leq q$. The projective plane ℓ_0 intersects Σ_{n-3}. In this case $\ell_1, \ldots, \ell_q$ are the only ideal lines which contain p_1 and have distance one from p_2.

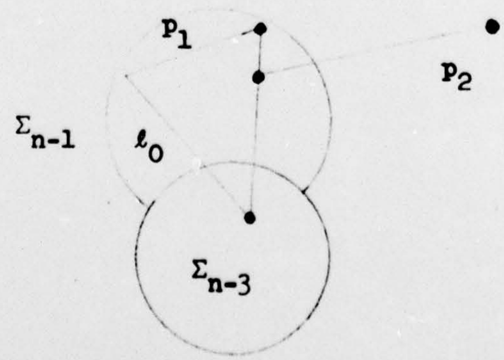

Figure 5

Case 2. The projective line p_2 intersects p_1 and is contained in Σ_{n-1} (Figure 6). The projective plane $\langle p_1, p_2 \rangle$ intersects Σ_{n-3} and hence is not an ideal line. Therefore $d(p_1, p_2) = 2$. Let ℓ_1 be any projective plane which contains p_1

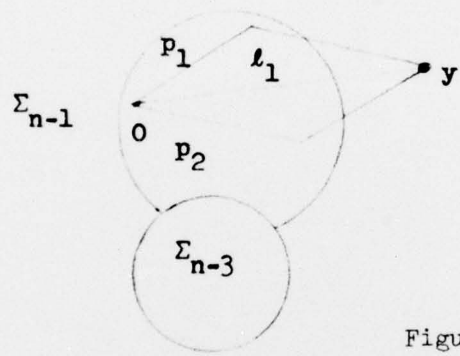

Figure 6

and is not contained in Σ_{n-1}. Then it is easily seen that $p_1 \subseteq \ell_1$ and $d(p_2, \ell_1) = 1$. In case 2 the number of ideal lines ℓ satisfying $p_1 \subseteq \ell$, $d(p_2, \ell) = 1$ is q^{n-2}. Therefore (f4) does not hold in π with respect to $(q - 1)$. Obviously π' is not a $(s, q - 1, d)$ - projective incidence structure for any choice of s and d.

This completes the proof of the minimality of the system of axioms (f1) - (f5).

Concluding Remarks. Consider a simple graph whose vertices are s-dimensional subspaces of a d-dimensional vector space V over $GF(q)$. Two vertices in this graph are adjacent iff the corresponding s-dimensional subspaces intersect in an (s-1)-dimensional subspace. This graph will be called → next page

an (s, q, d)-projective graph. The Theorem 1 of this paper can be used to obtain a characterization of the (s, q, d)-projective graphs provided d is larger than some function of s and q. ~~We are also considering~~ are also concerned characterization problems of Affine spaces and Polar spaces interms of flats of higher dimensions. These results will be communicated in a subsequent communication.

References.

1. R. C. Bose, Strongly Regular Graphs, Partial Geometries and Partially Balanced Designs, Pacific J. Math. 13 (1963), 389-419.

2. M. Hall., Jr., The Theory of Groups, Chapter 20, The MacMillan Company, New York, 1959.

3. A. J. Hoffman, $-1-\sqrt{2}$?, Proceedings of the Calgary International Conference on Combinatorial Structures and their Applications, Gordon and Breach, New York, 1970, 173-176.

4. C. C. Sims, On Graphs with Rank 3 Automorphism Group, Unpublished Manuscript.

5. Alan P. Sprague, A Characterization of Projective and Affine 3-schemes, Ph.D. thesis, The Ohio State University, 1973.